Math Help desk
Fourier Analysis

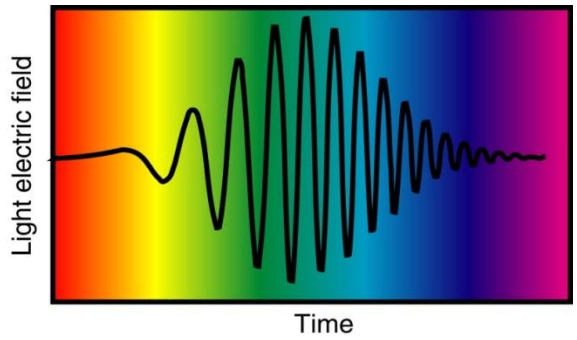
Department of Electrical engineering

OVERVIEW

- Introduction
- Fourier series
  - EVEN and ODD Functions
  - Orthogonality Principle (Proof)
  - Examples
  - Matlab
- Fourier Transform
  - Properties
  - Examples

Fourier Analysis

- All waveforms, what you observe in the universe, are actually just the sum of simple sinusoids of different frequencies.



- Fourier Series :The **Fourier Series** breaks down a periodic function into the sum of sinusoidal functions

- The Fourier Transform: is the mathematical tool that shows us how to deconstruct the waveform (non-periodic) into its sinusoidal components.

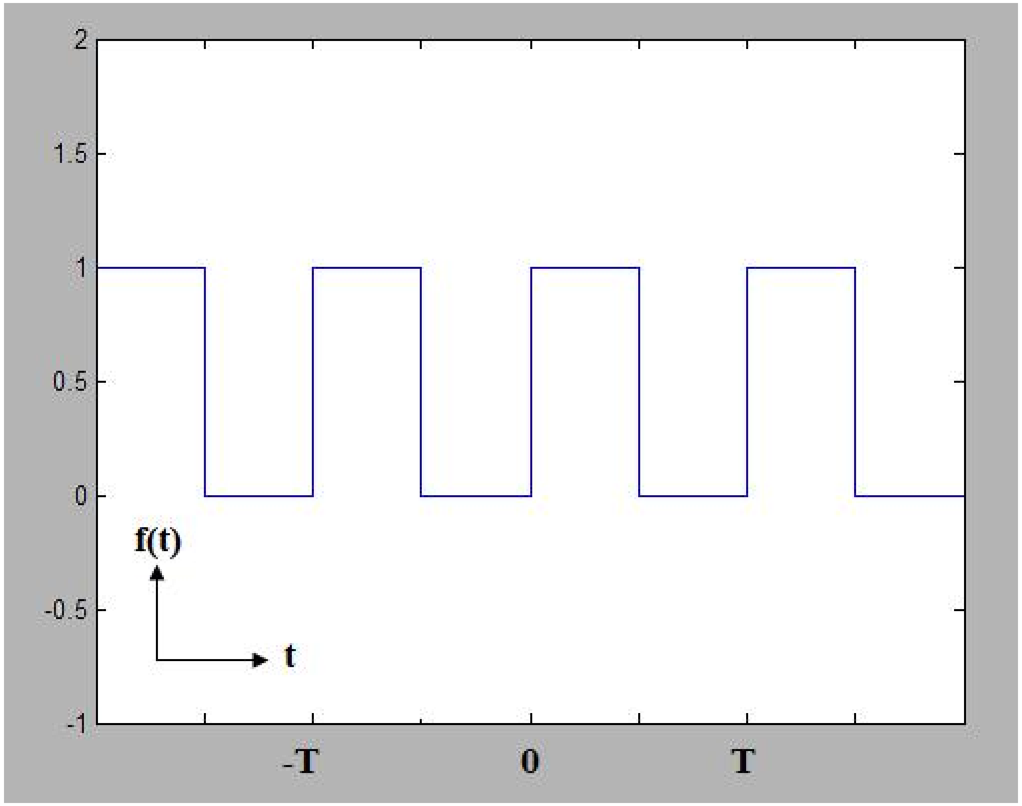
Fourier Series & Fourier Transform

- **Comparison of FT and FS**
  - Fourier Series: Used for periodic signals
  - Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signal)

	Synthesis	Analysis
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ Fourier Series	$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$ Fourier Coefficients
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ Inverse Fourier Transform	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ Fourier Transform

Fourier Series

- A function is periodic, with fundamental period T, if the following is true for all t:
- $f(t+T)=f(t)$



Fourier Series Representation

$$f(x) = a_0 + \sum_{n=1}^{n=\infty}\left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)\right)$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

Fourier series - Orthogonality

$$a_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\frac{2n\pi x}{T}\,dx$$

$$b_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin\frac{2n\pi x}{T}\,dx$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2n\pi x}{T}\right)\cos\left(\frac{2m\pi x}{T}\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2n\pi x}{T}\right)\sin\left(\frac{2m\pi x}{T}\right)dx = 0 \quad \text{for all } m, n$$

EVEN and ODD function

- **Even Functions**

A function $f(t)$ is said to be **even** ; if $f(-t)=f(t)$ for all values of t.

The graph of an **even** function is always symmetrical about the ***y*-axis.**

- **Odd Functions**

A function f(t) is called odd ; if f(-t)=-f(t)

- **If a periodic function is even then its Fourier series expansion contains only** cosine terms.
- **If a periodic function is odd ,its Fourier expansion contains only sine terms.**

Fourier Series

Examples (matlab)

SOLVED PROBLEMS

1.For a function defined by $f(x)=|x|, -\pi < x < \pi$ obtain a Fourier series

Solution

$f(x)=|x|$ is an even function

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

SOLUTION

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |x| dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2} \cos nx$$

PROBLEM 2

$$f(x) = \begin{cases} -k & when\ -3 < x < 0 \\ k & when\ 0 < x < 3 \end{cases}$$

Is the function even or odd. Find the Fourier series of f(x)

SOLUTION

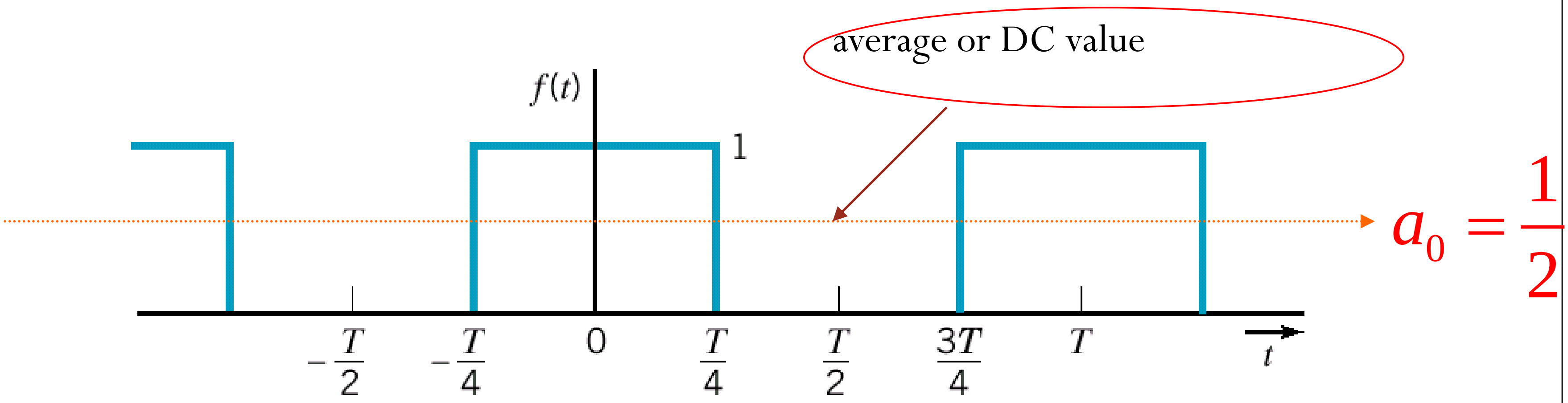
$f(x)$ is odd function

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 k\sin\frac{n\pi x}{3}\,dx$$

Example 15.3-1 determine Fourier Series and plot for N = 7



$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\int_{-T/4}^{T/4} 1\,dt = \frac{1}{2}$$

Example 15.3-1(cont.)

An *even function* exhibits symmetry around the vertical axis at $t = 0$ so that $f(t) = f(-t)$.

$$\therefore \quad b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} 1\sin n\omega_0 t\,dt = 0$$

Determine only a_n

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} 1\cos n\omega_0 t\,dt$$

$$= \frac{2}{T\omega_0 n}\sin n\omega_0 t\Big|_{-T/4}^{T/4}$$

Example 15.3-1(cont.)

$$a_n = \frac{1}{\pi n}\left[\sin\left(\frac{\pi n}{2}\right) - \sin\left(\frac{-\pi n}{2}\right)\right]$$

$$a_n = 0 \quad \text{when} \quad n = 2,\ 4,\ 6,\ \ldots$$

and

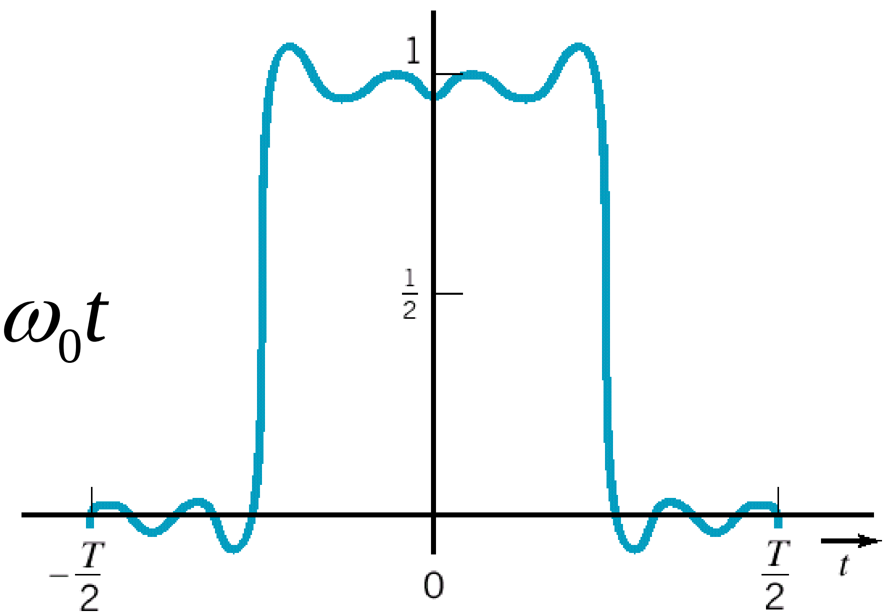
$$a_n = \frac{2(-1)^q}{\pi n} \quad \text{when} \quad n = 1,\ 3,\ 5,\ \ldots$$

where

$$q = \frac{(n-1)}{2}$$

$$f(t) = \frac{1}{2} + \sum_{n=1,odd}^{N} \frac{2(-1)^q}{\pi n} \cos n\omega_0 t$$

$$a_1 = \frac{2}{\pi}, a_3 = \frac{-2}{3\pi}, a_5 = \frac{2}{5\pi}, a_7 = \frac{-2}{7\pi}$$



$$= \frac{2}{3}\left[\frac{-k\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right]_0^3$$

$$= \frac{2k}{n\pi}[1-(-1)^n]$$

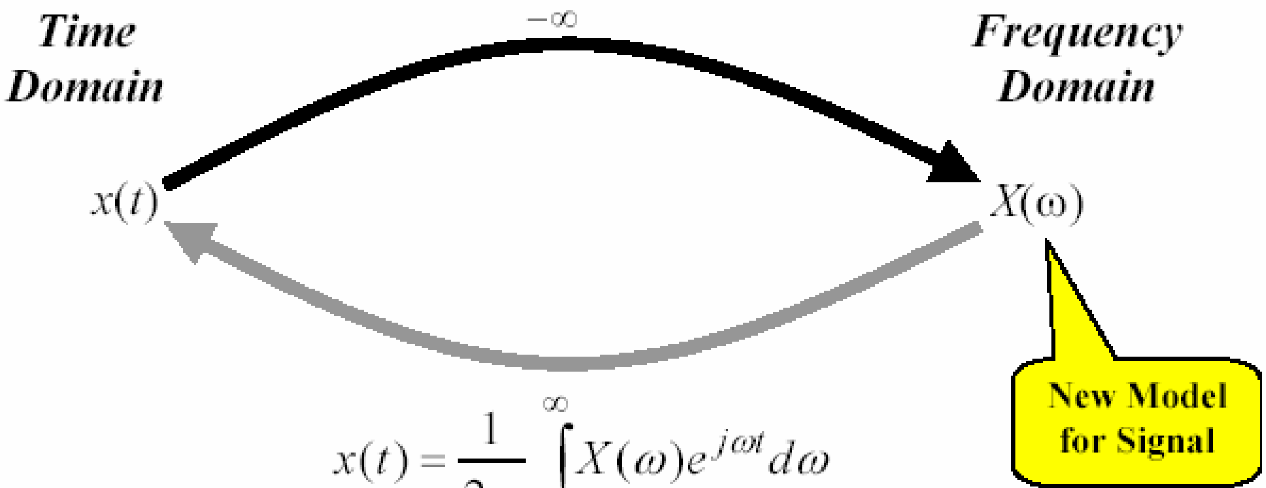
$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\frac{n\pi x}{3}$$

FOURIER TRANSFORM

Fourier Transform Viewpoint

View FT as a transformation into a new "domain"

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$x(t)$ is the "time domain" description of the signal

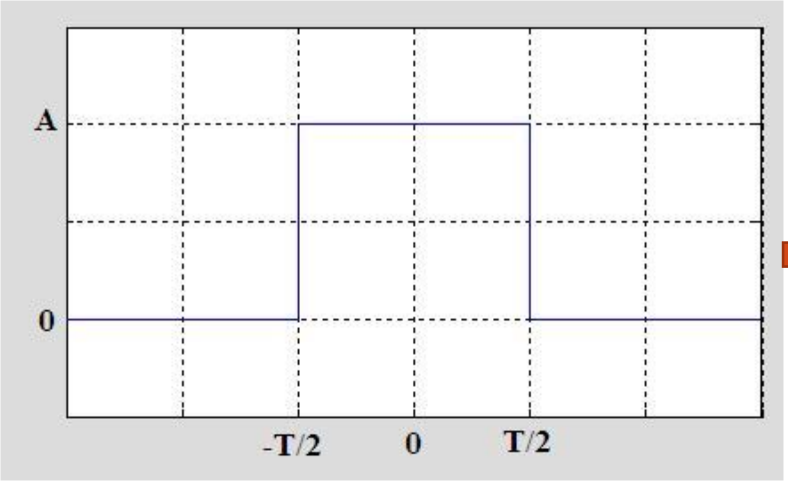
$X(\omega)$ is the "frequency domain" description of the signal

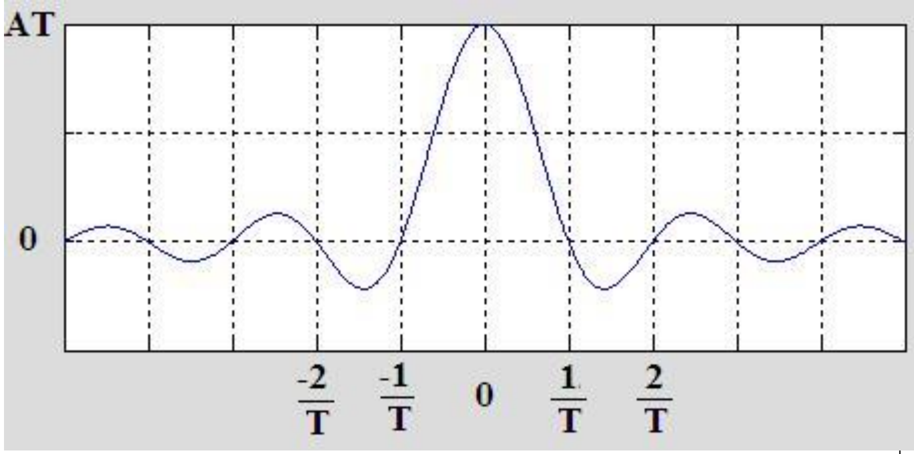
Fourier Transform Representation

- $f(t)$ lives in the **time domain**, and $F(\omega)$ lives in the **frequency domain**

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega$$



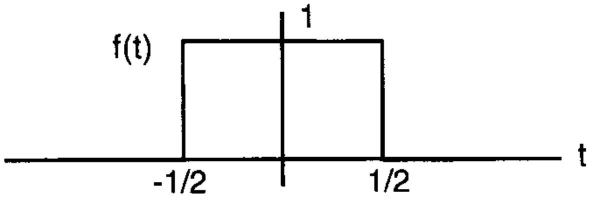


sinc function

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

FT of a rectangle function: rect(t)

$$F(\omega) \;=\; \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$



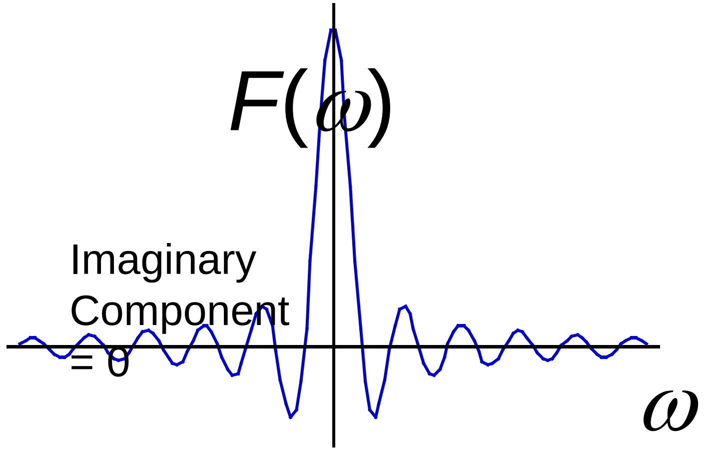
$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)dt = \frac{1}{-i\omega}\left[\exp(-i\omega t)\right]_{-1/2}^{1/2}$$

$$= \frac{1}{-i\omega}\left[\exp(-i\omega/2) - \exp(i\omega/2)\right]$$

$$= \frac{1}{(\omega/2)}\,\frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i}$$

$$= \frac{\sin(\omega/2)}{(\omega/2)}$$

$$F(\omega) = \mathrm{sinc}(\omega/2)$$



Fourier Transform

Examples

Fourier Transform

- Example: Fourier transform of $\sin\left(2\pi k_0 x\right)$

$$\therefore F_x\left(k\right) = \int_{-\infty}^{\infty} f\left(x\right) e^{-2\pi i k x} dx$$

Fourier Transform

- Example: Fourier transform of $\sin(2\pi k_0 x)$

$$\therefore F_x(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$$

$$F_x[\sin(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} \sin(2\pi k_0 x) e^{-2\pi i k x} dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{2\pi i k_0 x} - e^{-2\pi i k_0 x}}{2i} \right) e^{-2\pi i k x} dx$$

Fourier Transform

- Example: Fourier transform of $\sin(2\pi k_0 x)$

$$\therefore F_x(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$$

$$F_x[\sin(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} \sin(2\pi k_0 x) e^{-2\pi i k x} dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{2\pi i k_0 x} - e^{-2\pi i k_0 x}}{2i} \right) e^{-2\pi i k x} dx$$

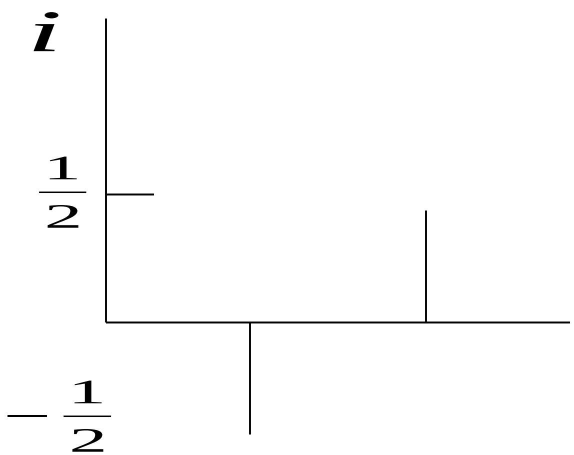
$$= \frac{i}{2} \int_{-\infty}^{\infty} \left(e^{-2\pi i (k+k_0)x} - e^{-2\pi i (k-k_0)x} \right) dx$$

Fourier Transform

- However the delta function is,

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi i k x} dk$$

$$\therefore F_x[\sin(2\pi k_0 x)](k) = \tfrac{i}{2}\left(\delta(k+k_0) - \delta(k-k_0)\right)$$



Proof of Convolution Property

- Taking Fourier transforms gives:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right) e^{-j\omega t} dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t} dt \right) d\tau$$

- By the time shift property, the bracketed term is $e^{-j\omega\tau}H(j\omega)$, so

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} H(j\omega)d\tau$$

$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau$$

$$= H(j\omega)X(j\omega)$$