Tutorial – Integral Transforms
EEM1026 ENGINEERING MATHEMATICS II

Laplace Transforms

1. Using definition only, find the Laplace transforms of
   (i) e^{-at}
   (ii) $cos\omega t$
   (iii) $sin\omega t$
   (iv) $at^2 + bt + c$
   (v) $sinh\, 3t + sin\, 3t$.

   answers: $\frac{1}{s+a}$, $\frac{s}{s^2+\omega^2}$, $\frac{\omega}{s^2+\omega^2}$, $\frac{2a}{s^3}+\frac{b}{s^2}+\frac{c}{s}$, $\frac{3}{s^2-9}+\frac{3}{s^2+9}$

   (b) Find the inverse transforms of [*Hint*: Partial fractions may be useful here]
   (i) $\frac{1}{(s+3)(s+7)}$
   (ii) $\frac{s-1}{(s-2)(s-3)(s-4)}$.

   answers: $\frac{1}{4}\left(e^{-3t}-e^{-7t}\right)$, $\frac{1}{2}e^{2t}-2e^{3t}+\frac{3}{2}e^{4t}$

2. Find the Laplace transforms of the following functions by using the ***s-Shifting Property***.
   (a) $0.4te^{2.5t}$
   (b) $e^t cost$
   (c) $e^{-3t}\left(cos2t-\frac{3}{2}sin2t\right)$.

   answers: $\frac{0.4}{(s-2.5)^2}$, $\frac{s-1}{s^2-2s+2}$, $\frac{s}{s^2+6s+13}$

3. By using the ***Derivative of Transforms***, find the Laplace transforms of:
   (a) te^t
   (b) $2t\, sinh\, t$
   (c) $t^2 cos\, \omega t$
   (d) $te^{-t}cosh\, 2t$.

   answers: $\frac{1}{(s-1)^2}$, $\frac{4s}{(s-1)^2}$, $\frac{2s\left(s^2-3\omega^2\right)}{\left(s^2+\omega^2\right)^3}$, $\frac{s^2+2s+s}{\left(s^2+2s-3\right)^2}$

4. Using the ***Integration of Transforms***, find $f(t)$ if $L\,[f(t)]$ is

   (a) $\ln\dfrac{s}{s-1}$

   (b) $\tan^{-1}\left(\dfrac{\omega}{s}\right)$.

answers: $\dfrac{1}{t}\left(e^{t}-1\right),\ \dfrac{1}{t}\sin\omega t$

5. By using the ***Transform of Derivatives***,

   (a) find (i) te^{t}

   (ii) $\sin^{2}\omega t$.

answers: $\dfrac{1}{(s-1)^{2}},\ \dfrac{2\omega^{2}}{s\left(s^{2}+4\omega^{2}\right)}$

   (b) show that $L\,(t\sin\omega t)=\dfrac{2\omega s}{\left(s^{2}+\omega^{2}\right)^{2}}$.

6. By using the ***Transform of Integrals***, find the inverse transforms of

   (a) $\dfrac{1}{s^{2}(s+1)}$

   (b) $\dfrac{54}{s^{3}(s-3)}$

answers: $t+e^{-t}-1,\ 2e^{3t}-9t^{2}-6t-2$

7. Using Laplace transform methods, solve the following differential equations, subject to the specified initial conditions:

   (a) $3y'-4y=\sin 2t$ $\quad y(0)=\dfrac{1}{3}$

answer: $y(t)=\dfrac{35}{78}e^{\frac{4t}{3}}-\dfrac{3}{26}\left(\cos 2t+\dfrac{2}{3}\sin 2t\right)$

   (b) $y''+2y'+y=4\cos 2t$ $\quad y(0)=0$ and $y'(0)=2$

answer: $y(t)=-\dfrac{12}{25}\cos 2t+\dfrac{16}{25}\sin 2t+\dfrac{12}{25}e^{-t}+\dfrac{6}{5}te^{-t}$

   (c) $\dfrac{d^{2}y}{dt^{2}}+2\dfrac{dy}{dt}+3y=3t$ $\quad y(0)=0$ and $y'(0)=1$

answer: $y(t)=-\dfrac{2}{3}+t+\dfrac{2}{3}e^{-t}\left[\cos\left(\sqrt{2}t\right)+\sqrt{\dfrac{1}{2}}\sin\left(\sqrt{2}t\right)\right]$

(d) $y'' + 8y' + 16y = 16\,sin\,4t \qquad y(0) = -\frac{1}{2}$ and $y'(0) = 1$

answer: $y(t) = te^{-4t} - \frac{1}{2}cos 4t$

(e) $y''' + y'' + y' + y = cos 3t \qquad y(0) = 0,\ y'(0) = 1$ and $y''(0) = 1$

answer: $y(t) = \frac{9}{20}e^{-t} - \frac{3}{80}sin 3t - \frac{1}{80}cos 3t + \frac{25}{16}sin t - \frac{7}{16}cos t$

8. Sketch the given functions and find their Laplace transforms using the ***t-Shifting Property***.
   (a) $4u(t-\pi)\,cos\,t$
   (b) $f(t) = \begin{cases} sin\,\omega t, & 0 \le t < \frac{\pi}{\omega} \\ 0, & t \ge \frac{\pi}{\omega}. \end{cases}$

answers: $-\frac{4e^{-\pi s}s}{(s^2+1)},\ \frac{\omega}{s^2+\omega^2}\left(1+e^{-\frac{\pi s}{\omega}}\right)$

9. (a) Compute the following convolutions $\left[f * g = \int_0^t f(\tau)g(t-\tau)d\tau \right]$,
   (i) $1 * 1$
   (ii) $1 * e^t$
   (iii) $u(t-\pi) * cos\,t$

answers: $t,\ -1+e^t,\ \begin{cases} 0, \text{if } t < \pi \\ -sin\,t, \text{if } t \ge \pi. \end{cases}$

   (b) Using the ***Convolution Theorem***, find the inverse transforms of the following s-domain functions:
   (i) $\frac{1}{s(s-2)}$
   (ii) $\frac{s}{(s^2+\omega^2)^2}$
   (iii) $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

answers: $\frac{1}{2}(e^{2t}-1),\ \frac{1}{2\omega}t\,sin\,\omega t,\ t\,cos\,\omega t$

10. Solve the following integral equations:

(a) $y(t) = \cos t + 2\int_0^t y(\tau)\cos(t-\tau)d\tau$

(b) $y(t) = 2t - 4\int_0^t y(\tau)(t-\tau)d\tau$

answers: $y(t) = e^t + te^t$, $y(t) = \frac{1}{2}\left(e^{4t} - 1\right)$

Fourier Transforms

11. Without the help of tables, find the Fourier transforms of the following functions:

(a) $f(t) = \begin{cases} e^t, & t < 0 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(t) = \begin{cases} 1 + \frac{t}{2}, & -2 < t < 0 \\ 1 - \frac{t}{2}, & 0 < t < 2 \end{cases}$

answers: $\frac{1}{\sqrt{2\pi}(1 - i\omega)}$, $\frac{1}{\sqrt{2\pi}}\frac{e^{-i\omega}(1 + i\omega) - 1}{\omega^2}$, $\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$, $\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega}{\omega}\right)^2$

12. Let $f(t) = te^{-t}$, $t > 0$. Find the Fourier transform of $f(t)$
    (a) using the Transform of Derivative property
    (b) using the Convolution Theorem [*Hint*: Begin by showing that $te^{-t} = e^{-t} * e^{-t}$]

answer: $\frac{1}{\sqrt{2\pi}(1 - i\omega)^2}$

13. Using Fourier transform methods, solve the following differential equations:

(a) $-y' + 3y = 30u(x)e^{-3x}, \quad (-\infty < x < \infty)$

$y(0) = 0,\ y(\infty)$ bounded, and $u(x)$ is the unit-step function.

answer: $y(x) = 5e^{-3|x|}$

(b) $y' + 9y = \delta(x), \quad (-\infty < x < \infty)$

$y(0) = 0,\ y(\infty)$ bounded. [*Hint*: $\mathrm{F}\{\delta(x)\} = \dfrac{1}{\sqrt{2\pi}}$]

answer: $y(x) = \begin{cases} e^{-9x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Z Transforms

14. Find the Z transform of the following sequences, $k = 0, 1, 2, \cdots$.

(a) 2^k

(b) $k2^k$

(c) k^2

answers: $\dfrac{z}{z-2}$, $\dfrac{2z}{(z-2)^2}$, $\dfrac{z(z+1)}{(z-1)^3}$.

15. Find the inverse z transform of the following functions.

(a) $\dfrac{z}{z+4}$

(b) $\dfrac{z}{(z+2)(z-1)}$

(c) $\dfrac{z^2}{(z-3)^2}$

answers: $(-4)^k$, $-\dfrac{1}{3}(-2)^k + \dfrac{1}{3}$, $(k+1)3^k$

16. Solve the following difference equations.

(a) $y_{k+1} - 3y_k = 0$, $y_0 = 4$.

(b) $y_{k+2} - 5y_{k+1} + 6y_k = 0$, $y_0 = 0$, $y_1 = 2$.

answers: $\{y_k\} = \{4(3)^k\}$, $\{y_k\} = \{2(3)^k - 2(2)^k\}$.